I. Transformations :

In this section some of the basic transformations are discussed in detail.

1. Transformation $w = az+b$

soln :

$w = az+b$ is the composition of $w = az$ and $w = z+c$.

Let $\quad w = u+iv$, $z = x+iy$ and $b = b_1 + ib_2$

Then $\quad w = x+iy + (b_1 + ib_2)$

$$= (x+b_1) + i(y+b_2)$$

Thus under this transformation the set of Points is shifted through a distance $|b|$ in the direction specified by arg b.

Consider $w = az$ where a is a non-zero Complex constant.

Let $z = re^{i\theta}$ and $a = r_1 e^{i\theta_1}$

Then $\quad W = r_1 e^{i\theta_1} re^{i\theta}$

$$= rr_1 e^{i(\theta+\theta_1)}$$

Thus the transformation consists of a magnification (or) contraction of the radius vector by the factor $|a|$ and a rotation through an angle θ_1 about the origin.

So $w = az+b$ is a rotation and magnification or contraction about the origin and then a translation.

2. Transformation $w = \frac{1}{z}$

Soln:

The transformation is conformal for all z except $z=0$, since $\frac{dw}{dz} = \frac{-1}{z^2}$ and is defined for all z except $z=0$.

Now, $w = u+iv = \frac{1}{x+iy}$

$= \frac{x-iy}{x^2+y^2}$

ie) $u = \frac{x}{x^2+y^2}$; $v = \frac{-y}{x^2+y^2}$

Also $u^2+v^2 = \frac{x^2+y^2}{(x^2+y^2)^2}$

$= \frac{1}{x^2+y^2}$

$x^2+y^2 = \frac{1}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$; $y = \frac{-v}{u^2+v^2}$

(i) The circle with centre at origin in the z-plane are of the form $x^2+y^2 = k$. These circle transform in to $u^2+v^2 = \frac{1}{k}$ in the w-plane.

ie) Unit circle with centre at the origin in the z-plane transform in to unit circle in the w-plane with centre at the origin.

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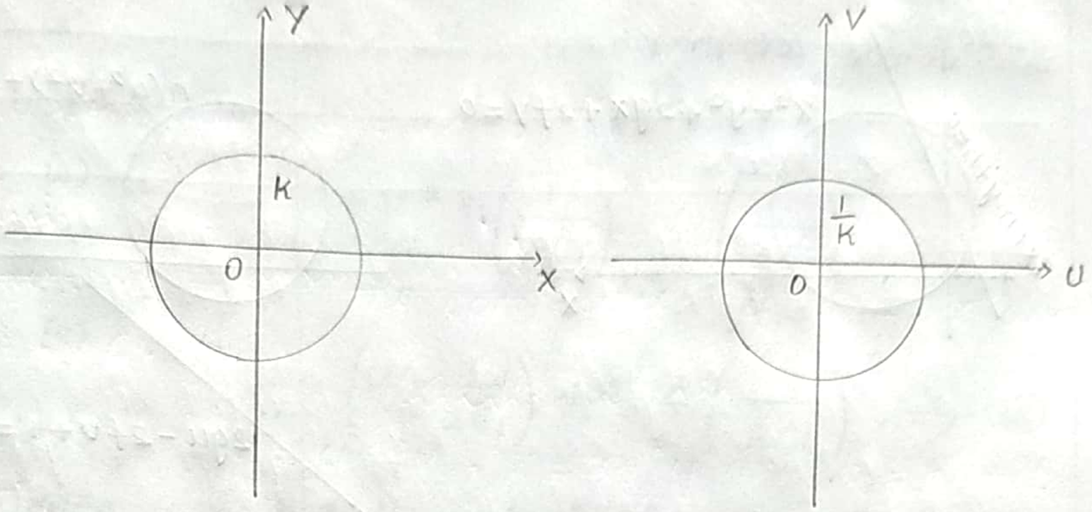
Also, circle lying outside unit circle in the z-Plane transform into circle with in the unit circle in the w-Plane.



z - Plane   w - Plane

(ii) Any circle in the z-Plane is of the form $x^2+y^2+2gx+2fy+c=0$.

This transforms in to

$$\frac{1}{u^2+v^2} + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

ie) $$c(u^2+v^2) + 2gu - 2fv + 1 = 0$$

Case (i) If $c \neq 0$, the circle in z-Plane transform in to circle in the w-Plane.

Case (ii) If $c = 0$, then circle in the z-Plane Passes through the origin get the transforms in to a straight line $2gu - 2fv + 1 = 0$ in the w-Plane.

(iii) A line in the z-Plane of the form $lx + my + n = 0$, transforms in to

$$l\frac{u}{u^2+v^2} - \frac{mv}{u^2+v^2} + n = 0$$

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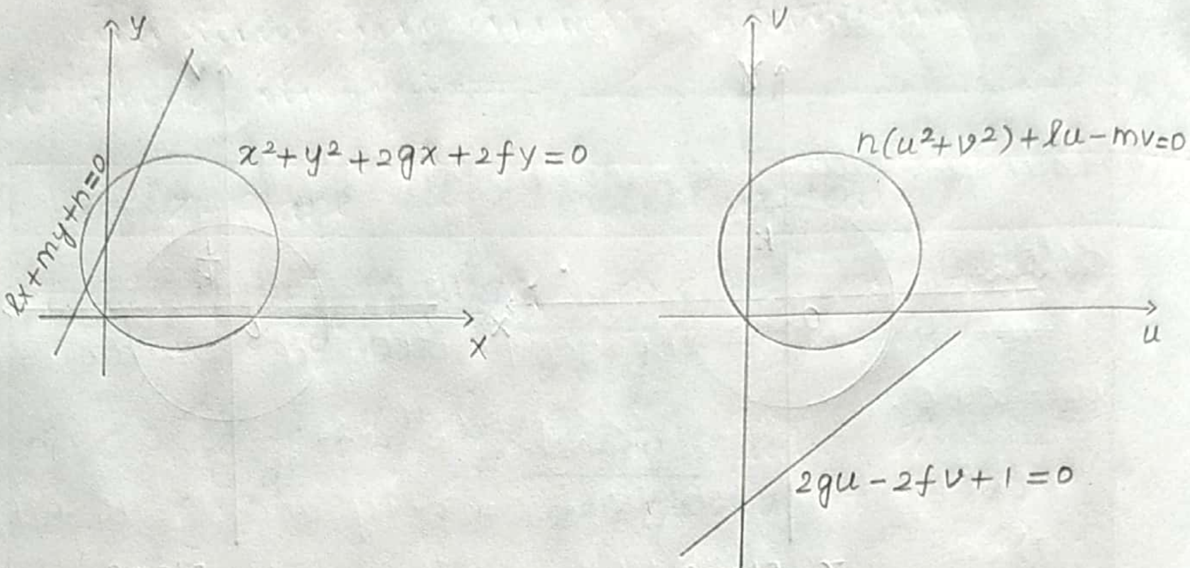
i.e) $n(u^2+v^2) + lu - mv = 0$

i.e) a circle Passing through the origin.



Hence the line in the z-Plane transform in to circle Passing through the origin in the w-plane. If $n=0$, the straight line in the z-Plane passes through the origin, transform in to the straight line through the origin $lu-mv=0$ in the w-plane.

(iv) The line $x=k$ transforms in to the circle

$$u^2+v^2-\frac{u}{k}=0$$

* The v-axis is a tangent to the circle at the origin.
* The y-axis [i.e., $x=0$] transforms in to the v-axis.
* The line $y=k_1$, transform in to the circle

$$u^2+v^2+\frac{v}{k_1}=0$$

* The u-axis touches this circle at the origin.
* The x-axis, [i.e., $y=0$] transforms in to the u-axis.

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* The half plane $x > c_1$ has its image in the region.

$$\frac{u}{u^2+v^2} > c_1 \quad \text{—— ①}$$

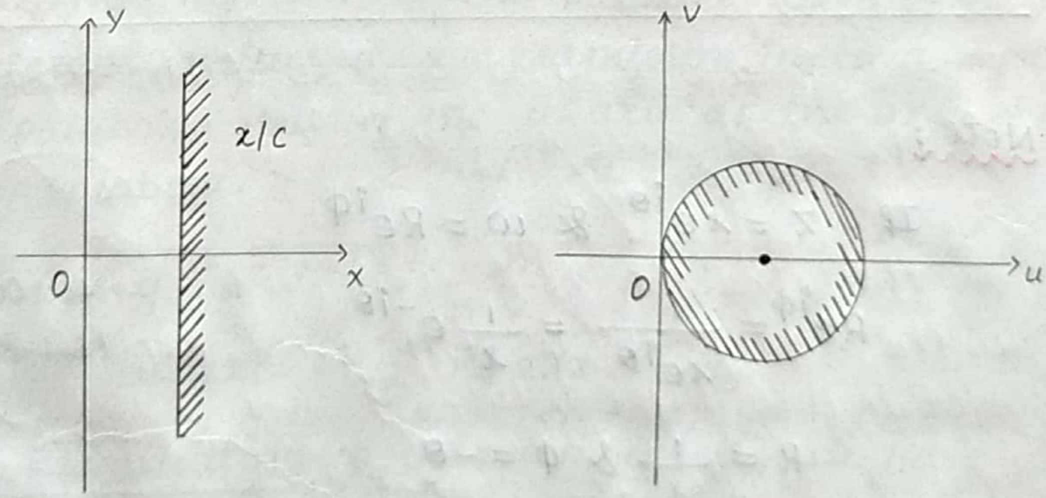
ie) $$u^2+v^2 < \frac{u}{c_1} \quad \text{when } c_1 > 0$$

ie) $$u^2 - \frac{u}{c_1} + v^2 + \left(\frac{1}{2c_1}\right)^2 < \left(\frac{1}{2c_1}\right)^2$$

ie) $$\left(u - \frac{1}{2c_1}\right)^2 + v^2 < \left(\frac{1}{2c_1}\right)^2 \quad \text{—— ②}$$

Hence the point w is inside a circle having it's center at $\left(\frac{1}{2c_1}, 0\right)$ and radius $\frac{1}{2c_1}$.

The v-axis is the tangent to this circle at the origin.



z-Plane　　　　　　w-Plane

Conversely, whenever u & v satisfy the inequality & $c_1 > 0$, then the inequality (1) follows and therefore $x > c_1$.

Consequently, every point inside the circle is the image of some point in the half plane.

Also, the image of the half plane is the interior of the circular region.

The above transformation yields the following result:

z - Plane	w - Plane
(i) Circle not through the origin	(i) circle not through the origin.
(ii) Circle through the origin	(ii) Straight line not through the origin.
(iii) Straight line not through the origin	(iii) Circle through the origin
(iv) Straight line through the origin	(iv) Straight line through the origin.

Note:

If $z = re^{i\theta}$ & $w = Re^{i\phi}$

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

$$R = \frac{1}{r} \ \& \ \phi = -\theta$$

ie) $\quad Rr = 1.$

Hence this transformation is equivalent to an inversion in unit circle and reflection the real axis.

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3. Transformation $w = z^2$

soln:

This transformation $w = z^2$ is analytic and so representation is conformal everywhere except when $f'(z) = 0$ when $z = 0$.

Now, $u + iv = w = (x+iy)^2$

$$= x^2 + i^2 y^2 + i2xy$$

$$= (x^2 - y^2) + i(2xy)$$

$\therefore u = x^2 - y^2 \ ; \ v = 2xy$

(i) Let $x = k$

$$u = k^2 - y^2 \ \& \ v = 2ky$$

Eliminating y between u & v, we get $v^2 = 4k^2(k^2 - u)$, which is a Parabola.

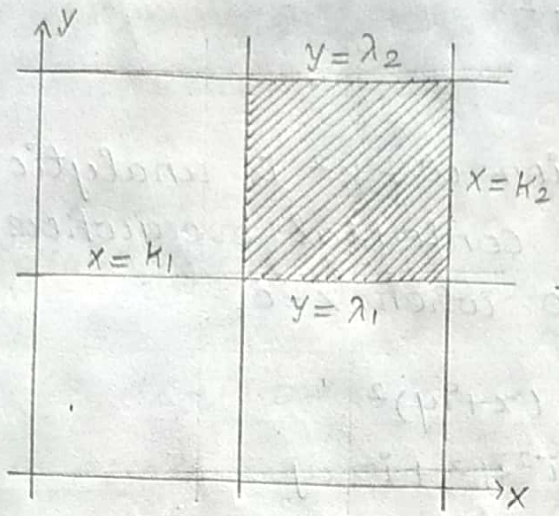
ie) The system of the Parallel lines $x = k$ for different values of k, transform in to a system of Parabolas having the u-axis as the axis of the Parabola.

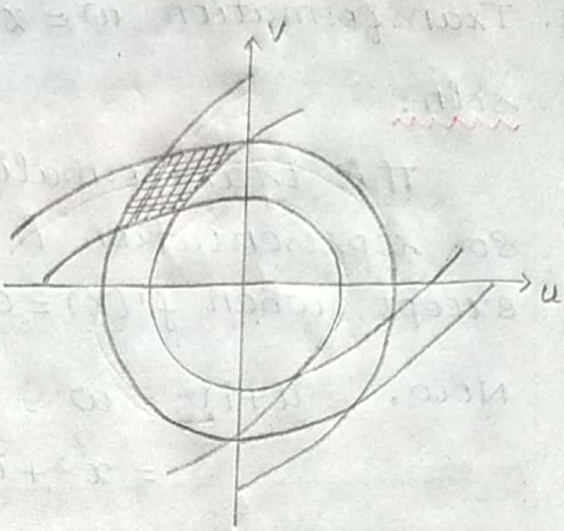
(ii) Let $y = \lambda$

$$u = x^2 - \lambda^2 \ ; \ v = 2\lambda x$$

Eliminating x between u & v, we have $v^2 = 4\lambda^2(u + \lambda^2)$ which is a Parabola.

Hence the system of lines $y = \lambda$ maps in to the family of Parabolas $v^2 = 4\lambda^2(u + \lambda^2)$ having the u-axis as the axis of the Parabolas.





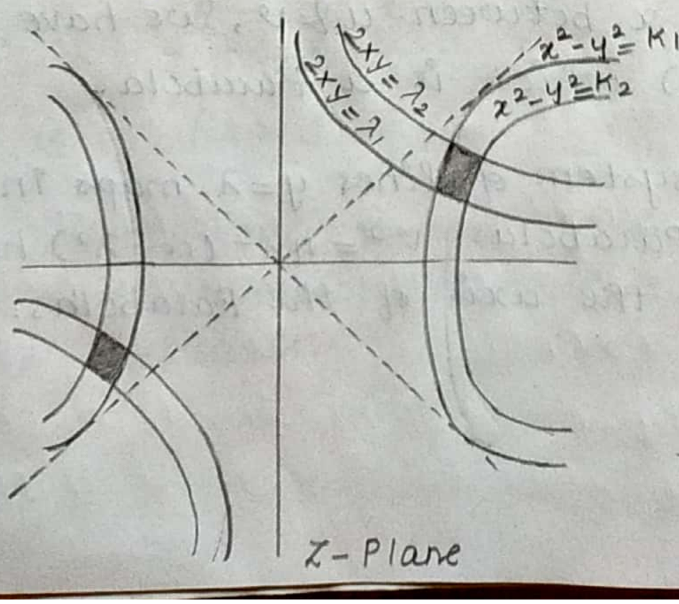
The two families of the above Parabolas are orthogonal since families of lines $x=k$ and $y=\lambda$ are orthogonal of the z-Plane.

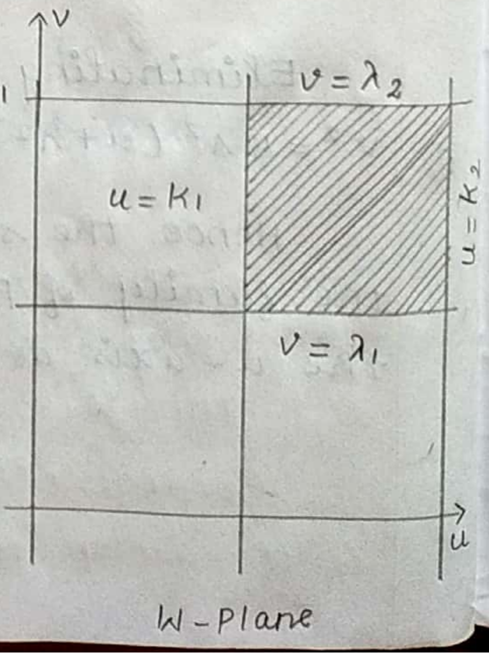
We also note that the two families of Parabolas are confocal and the common focus is the origin.

The image of $x=0$ is $v=0$ (with $u<0$)
ie) The left half real axis.

The image of $y=0$ is $v=0$ (with $u>0$)
ie) The right half real axis.

(iii) The straight line $u=k$ and $v=\lambda$ on the w-Plane map in w orthogonal families of rectanglar hyperbolas.





These two families of rectangular hyperbolas have the same centre (0,0) and the axes of the family of rectangular hyperbolas are the asymptotes of the other family of rectangular hyperbolas.

(iv) Let $z = re^{i\theta}$

Then $w = (re^{i\theta}) = r^2 e^{2i\theta} = \lambda e^{i\phi}$

$\lambda = r^2 \quad \phi = 2\theta.$

If z describes an arc of a circle of radius r subtending an angle θ at (0,0) then w describes an arc of circle of radius r^2 subtending an angle 2θ at it's origin.

ie) The angle at the origin are doubled under the mapping.

∴ Upper half of the z-Plane maps in to whole of w-Plane and first quadrant of z-Plane maps into upper half of the w-Plane.

Evidently, +ve x-axis maps into the +ve u-axis and +ve y-axis maps in to the -ve u-axis.

(v) Consider the circle with center at a and radius c, (a, c being real) in the z-Plane.

ie) $|z-a| = c \Rightarrow z - a = ce^{i\theta}.$

$$w = z^2 = (a+ce^{i\theta})^2 = a^2 + c^2 e^{2i\theta} + 2ace^{i\theta}$$

$w - a^2 + c^2 = c^2 + c^2 e^{2i\theta} + 2ace^{i\theta}$ (Add c^2 on both side)

$$= c^2(1+e^{2i\theta}) + 2ace^{i\theta}$$

$$= c^2(e^{i\theta}.e^{i\theta}) + 2ace^{i\theta}$$

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$$= c^2 e^{i\theta}(e^{i\theta} + e^{-i\theta}) + (2ac)e^{i\theta}$$

$$= 2ce^{i\theta}(a + c\cos\theta) \quad (\because 2\cos\theta = e^{i\theta} + e^{-i\theta})$$

Let $w - a^2 + c^2 = Re^{i\theta}$ Then Pole in the w-Plane is $w = a^2 - c^2$, in the Polar equation of the image when A is chosen as the Pole & R, θ as Polar co-ordinates.

$$Re^{i\theta} = 2ce^{i\theta}[a + c\cos\theta]$$

$$R = 2c[a + c\cos\theta]$$

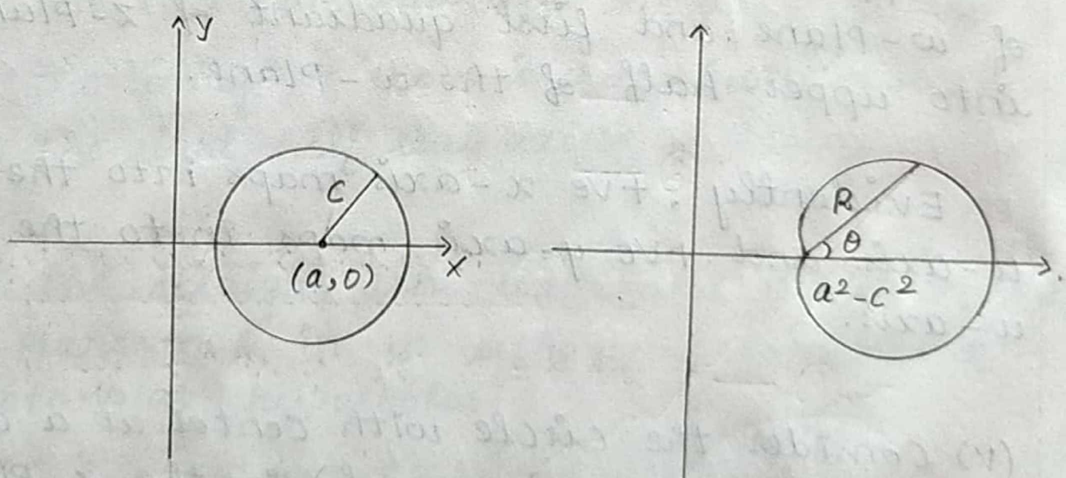
Which is an equation of limacon.

in Particular, if $a = c$,

$$R = 2a[a + a\cos\theta]$$

$$= 2a^2(1 + \cos\theta)$$

Which is an equation of a Cardioid.



Z - Plane $\qquad$ W - Plane

4. Transformation $w = \sqrt{z}$

soln:

$$w = \sqrt{z} \Rightarrow w^2 = z$$

By the inverse transformation $z = w^2$, the upper half w plane gives on to the entire z-plane once.

Now, $x + iy = (u+iv)^2$

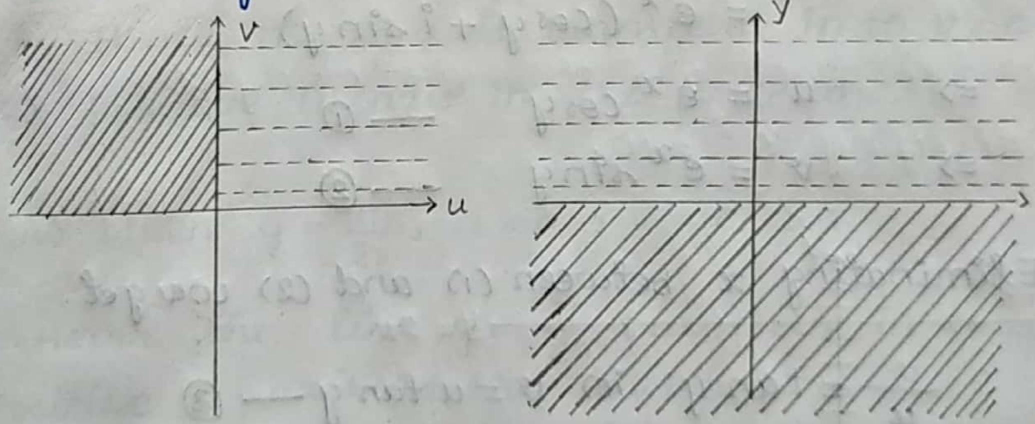
$$= u^2 - v^2 + i2uv$$

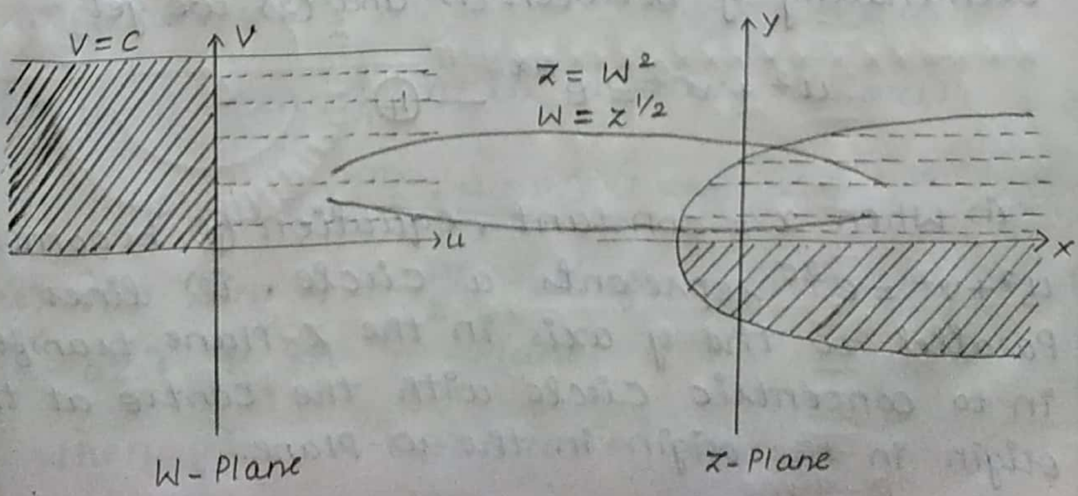
$$x = u^2 - v^2 \; ; \; y = 2uv$$

(i) Let $v = c$

Then $x = u^2 - c^2 \; ; \; y = 2uc$

Eliminating u from x & y, we get $y^2 = 4c^2(x + c^2)$ which is a parabola, when $c = 0$, $v = 0$ goes on the right semi real axis.





The infinite strip $0 \leq v \leq c$ goes on to the interior of the parabola $y^2 = k^2(x+c^2)$ under the transformation $z = w^2$. But under the transformation $w = z^{1/2}$ the upper half of the interior of the parabola $y^2 = k^2(x+c^2)$ goes on to the infinite strip $0 \leq u \leq \infty, 0 \leq u \leq c$ because the arguments of the points in the upper half of the interior of the parabola vary from 0 to π and the arguments of the points in the infinite strip $0 \leq u \leq \infty$, $0 \leq v \leq c$ vary from 0 to $\frac{\pi}{2}$.

5. Transformation $w = e^z$

soln:

$$w = u+iv = e^{x+iy} = e^x . e^{iy}$$

$$= e^x(\cos y + i\sin y)$$

$$\Rightarrow \quad u = e^x \cos y \quad \text{--- (1)}$$

$$\Rightarrow \quad v = e^x \sin y \quad \text{--- (2)}$$

Eliminating x between (1) and (2) we get

$$\frac{v}{u} = \tan y \quad \text{ie) } v = u \tan y \text{ --- (3)}$$

Eliminating y between (1) and (2) we get

$$u^2 + v^2 = e^{2x} \quad \text{--- (4)}$$

(i) When x = constant, equation (4) becomes $u^2+v^2 = e^{2c}$ represents a circle. ie) lines parallel to the y axis in the z-plane transform in to concentric circle with the centre at the origin in the w-plane.

$x = k$ $(k>0)$ transform in to a concentric circle outside the unit radius circle.

$x = -k$ $(k>0)$ transform in to a concentric circle with in the unit radius circle.

(ii) When y = constant, equation ③ represents a line through the origin in the w-plane.

Hence the line parallel to the x-axis transform in to radial lines in the w-plane.

(iii) Line $y=a$, $y=b$ when $b-a < 2\pi$ corresponds to a wedge in the w-plane bounded by the radial lines whose inclination are a and b.

When $y=0$, from ① & ② $u=e^x$ and $v=0$.

Since e^x is always positive we have $u>0$ and $v=0$.

Hence the x-axis transform in to $v=0$.

ie) Positive u-axis in the w-plane.

(iv) When $y=\frac{\pi}{2}$, $u=0$, $v=e^x>0$

Hence the line $y=\frac{\pi}{2}$ transform in to positive v-axis in the w-plane.

(v) When $y=\pi$, $v=0$ and $u=-e^{-x}<0$.

ie) $y=\pi$ transform in negative u-axis.

When $y=\frac{3\pi}{2}$, $u=0$, $v=-e^{-x}<0$.

ie) the line $y=\frac{3\pi}{2}$ transform in to the negative v-axis in the w-plane.

When $y=2\pi$, $v=0$, $u=e^x>0$

Hence the line $y = 2\pi$ transforms into the Positive side of the u-axis in the ω-Plane.

Hence any horizontal strip of the z-Plane of height 2π will cover the entire ω-Plane once.

The strip $0 \le y \le 2\pi$ in the z-Plane is mapped on to $0 \le \arg \omega < 2\pi$ which is the whole of the ω-Plane excluding the two points $\omega = 0$ & $\omega = \infty$, since the arguments of 0 & ∞ are not defined.

But $e^{x+iy+2n\pi i} = e^{x+iy} \cdot e^{2n\pi i} = e^{x+iy}$, $n = 0, \pm 1, \pm 2, \ldots$ so each of the following strip in the z-Plane.

$-2\pi \le y < 0,\ 0 \le y < 2\pi,\ 2\pi \le y < 4\pi \ldots\ldots$

goes on to z-Plane once.

ie) for each ω, there are many z's satisfying $e^z = \omega$.